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Covering numbers under small perturbations☆

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Abstract

We investigate how the covering number of the elements can change under arbitrarily small perturbations. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $\mathbb{F}$ denote the field of real numbers or the field of complex numbers. Let $(v_1, \ldots, v_m)$ be a family of vectors of $\mathbb{F}^n$. To each vector v_i in $(v_1, \ldots, v_m)$ one associates a real number called the covering number of v_i in $(v_1, \ldots, v_m)$ and usually denoted by $s_{v_i}(v_1, \ldots, v_m)$. The notion of covering number of an element was introduced in [3] and studied in [3,5,6,7].

The aim of this article is to characterize the covering number of an element that can be obtained with arbitrarily small perturbations of the vectors $v_1, \ldots, v_m$. This characterization will be studied in Section 3. In [8] it was studied how the rank partitions can change under arbitrarily small perturbations.

In Section 2, we introduce concepts and notations necessary for Section 3. In Section 4, we establish the relationship between the concepts introduced in Section 2 and the corresponding concepts in matroid theory and we present some results needed in Section 5. Finally, in Section 5, we give the proofs of all results stated in Section 3.

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2. Preliminaries

Let m be a positive integer. A partition of m is a sequence of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_r)$ satisfying

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_r$$

and

$$\sum_{i=1}^{r} \alpha_i = m.$$

If $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition of m and $\alpha_r > 0$ we say that r is the length of α (i.e. the greatest k such that $\alpha_k > 0$). We do not distinguish between two partitions that differ only in the number of trailing zeros. In particular, we can consider a partition α of m as an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ (by removing or adding a string of zeros).

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a partition of m. Suppose that n_1 terms of α are equal to 1, n_2 terms of α are equal to 2 and, in general, n_i terms of α are equal to i, $i = 1, \ldots, m$. We use also, as notation for the partition α, the finite sequence

$$\alpha = (m^{n_m}, (m-1)^{n_{m-1}}, \ldots, 1^{n_1}).$$

If $n_i = 0$, then i^{n_i} is usually left out.

Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers.

Let I be a finite set. A family $(v_i)_{i \in I}$ of vectors of $\mathbb{F}^n$ is a mapping $i \to v_i$ from I into $\mathbb{F}^n$. The cardinality of the family $(v_i)_{i \in I}$ is the cardinality of I. If $J \subseteq I$, the family $(v_i)_{i \in J}$ is a subfamily of $(v_i)_{i \in I}$; this subfamily is defined by the restriction to J of $i \to v_i$.

Let k be a positive integer. A subfamily $(v_i)_{i \in J}$ of $(v_i)_{i \in I}$ is k-independent if it is the union of k subfamilies each of which is linearly independent, that is, if there exist subsets $I_1, \ldots, I_k$ such that

1. $J = I_1 \cup \cdots \cup I_k$.
2. $(v_i)_{i \in I_j}$ is linearly independent, $j = 1, \ldots, k$.

The k-dimension of $(v_i)_{i \in I}$ is the maximum cardinality of the k-independent subfamilies of $(v_i)_{i \in I}$. As a subfamily is 1-independent if and only if it is linearly independent, the 1-dimension of $(v_i)_{i \in I}$ is the dimension of $\langle v_i : i \in I \rangle$, where $\langle v_i : i \in I \rangle$ denotes the subspace spanned by $(v_i)_{i \in I}$.

A subfamily $(v_i)_{i \in J}$ of $(v_i)_{i \in I}$ is a k-basis of $(v_i)_{i \in I}$ if it is k-independent and $|J|$ is equal to the k-dimension of $(v_i)_{i \in I}$.

If $I = \{1, \ldots, m\}$, the family $(v_i)_{i \in I}$ is simply denoted by $v_1, \ldots, v_m$. The k-dimension of $v_1, \ldots, v_m$ is denoted by $d_k(v_1, \ldots, v_m)$. By convention, $d_0(v_1, \ldots, v_m) = 0$.

Let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$. It was proved in [2] that the sequence

$$(d_1(v_1, \ldots, v_m) - d_0(v_1, \ldots, v_m), \ldots, d_m(v_1, \ldots, v_m) - d_{m-1}(v_1, \ldots, v_m))$$

is a partition of $d_m(v_1, \dots, v_m)$ and $d_m(v_1, \dots, v_m)$ is equal to the number of nonzero vectors in the family $v_1, \dots, v_m$. This partition is called the rank partition of $v_1, \dots, v_m$ and is denoted by

$$\rho(v_1, \dots, v_m).$$

Let $(v_i)_{i \in I}$ be a family of vectors of $\mathbb{F}^n$ and let r be a positive integer. A subfamily $(v_i)_{i \in J}$ of $(v_i)_{i \in I}$ is called an r-transversal of $(v_i)_{i \in I}$ if there exist pairwise disjoint subsets $J_1, \dots, J_r$ of J such that

1. $J = J_1 \cup \cdots \cup J_r$.
2. $(v_i)_{i \in J_s}$ is a basis of $\langle v_i : i \in J \rangle$, $s = 1, \dots, r$.

It is easy to conclude that a subfamily $(v_i)_{i \in J}$ is an r-transversal of $(v_i)_{i \in I}$ if and only if is r-independent and $|J| = r \cdot \dim(\langle v_i : i \in J \rangle)$.

It was proved in [1] that, if $(v_i)_{i \in J}$ and $(v_i)_{i \in J'}$ are maximal r-transversals of $(v_i)_{i \in I}$ (for the inclusion order), then

$$\langle v_i : i \in J \rangle = \langle v_i : i \in J' \rangle.$$

Let $v_1, \dots, v_m$ be a family of vectors of $\mathbb{F}^n$ and let v_i be a nonzero vector of $v_1, \dots, v_m$. The smallest integer s such that

$$d_s(v_1, \dots, v_m) > d_s(v_j : j \in \{1, \dots, m\} \setminus \{i\})$$

is called the covering number of v_i in $v_1, \dots, v_m$ and is denoted by

$$s_{v_i}(v_1, \dots, v_m)$$

or simply by s_i. By convention, if v_i is the zero vector, then $s_i = s_{v_i}(v_1, \dots, v_m) = m + 1$. The notion of covering number of an element was introduced in [3]. It is easy to conclude that if s_i is the covering number of v_i and $k < s_i$, then there exists a k-basis of $v_1, \dots, v_m$ which v_i does not belong.

Example 2.1. Let e_1, e_2 be a linearly independent family of vectors of $\mathbb{F}^n$ and let 0 be the zero vector of $\mathbb{F}^n$. Let $m = 6$ and define

$$v_1 = e_1, \quad v_2 = 0, \quad v_3 = e_1, \quad v_4 = e_2, \quad v_5 = e_1 - e_2, \quad v_6 = e_1.$$

Since $(v_i)_{i \in \{1,4,5,6\}}$ is a 2-basis of $v_1, v_2, v_3, v_4, v_5, v_6$; a 2-basis of v_1, v_2, v_4, v_5, v_6 and a 3-basis of v_1, v_2, v_4, v_5, v_6, then

$$d_2(v_1, v_2, v_3, v_4, v_5, v_6) = 4 = d_2(v_1, v_2, v_4, v_5, v_6),$$
$$d_3(v_1, v_2, v_3, v_4, v_5, v_6) = 5 > d_3(v_1, v_2, v_4, v_5, v_6) = 4$$

and

$$s_3 = s_{v_3}(v_1, \dots, v_6) = 3.$$

By convention, $s_2 = s_{v_2}(v_1, \dots, v_6) = 6 + 1 = 7$.

3. Small perturbations

Let $\| \ \|$ be a norm in $\mathbb{F}^n$ and let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$. Denote by $\mathscr{R}(v_1, \ldots, v_m)$ the set of all integers r such that, for every $\delta > 0$, there exist $v'_1, \ldots, v'_m \in \mathbb{F}^n$ such that

$$\|v'_j - v_j\| \leqslant \delta, \quad j \in \{1, \ldots, m\} \text{ and } \dim\langle v'_1, \ldots, v'_m\rangle = r.$$

The following result is elementary.

Lemma 3.1. *An integer r is in $\mathscr{R}(v_1, \ldots, v_m)$ if and only if*

$$\dim\langle v_1, \ldots, v_m\rangle \leqslant r \leqslant \min\{m, n\}.$$

In [8] it was stated a generalization of this result.

For every positive integer k, denote by $\mathscr{R}_k(v_1, \ldots, v_m)$ the set of all integers r such that, for every $\delta > 0$, there exist $v'_1, \ldots, v'_m \in \mathbb{F}^n$ such that

$$\|v'_j - v_j\| \leqslant \delta, \quad j \in \{1, \ldots, m\} \text{ and } d_k(v'_1, \ldots, v'_m) = r.$$

Theorem 3.2. *An integer r is in $\mathscr{R}_k(v_1, \ldots, v_m)$ if and only if*

$$d_k(v_1, \ldots, v_m) \leqslant r \leqslant \min\{m, kn\}.$$

The main purpose of this section is to obtain a similar result for the covering number of an element of $v_1, \ldots, v_m$ (Theorems 3.5 and 3.6).

Let v_i be an element of $v_1, \ldots, v_m$ and let $s_i = s_{v_i}(v_1, \ldots, v_m)$. Denote by $\mathscr{S}_i(v_1, \ldots, v_m)$ the set of all integers r such that, for every $\delta > 0$, there exist $v'_1, \ldots, v'_m \in \mathbb{F}^n$ such that

$$\|v'_j - v_j\| \leqslant \delta, \quad j \in \{1, \ldots, m\} \text{ and } s_{v'_i}(v'_1, \ldots, v'_m) = r.$$

Remark. Since $s_i \in \mathscr{S}_i(v_1, \ldots, v_m)$, then $\mathscr{S}_i(v_1, \ldots, v_m) \neq \emptyset$.

Proposition 3.3. *$m + 1 \in \mathscr{S}_i(v_1, \ldots, v_m)$ if and only if v_i is the zero vector of $\mathbb{F}^n$.*

Let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$. Let $(\rho_1, \ldots, \rho_m)$ be the rank partition of $v_1, \ldots, v_m$. For each v_i in $v_1, \ldots, v_m$ define

$$s_i = s_{v_i}(v_1, \ldots, v_m),$$
$$a_i = \min(\min\{j : n > \rho_j\}, s_i)$$

and

$$b_i = \max_{i \in J \subseteq \{1,\ldots,m\}} \left\{ \left\lceil \frac{|J|}{\max\{d_1(v_j : j \in J), 1\}} \right\rceil : \right.$$
$$\left. (\langle (v_j)_{j\in J}\rangle \cap (v_j)_{j\in\{1,\ldots,m\}}) = (v_j)_{j\in J} \right\},$$

where $\lceil x \rceil$ denotes the least integer greater or equal to x.

Example 3.4. Let e_1, e_2, e_3, e_4 be a basis of $\mathbb{F}^4$ and let 0 be the zero vector of $\mathbb{F}^4$. Let $m = 8$ and define

$$\begin{aligned} &v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_2, \quad v_4 = e_2, \\ &v_5 = 0, \quad v_6 = e_3, \quad v_7 = e_1 + e_3, \quad v_8 = e_4. \end{aligned}$$

Then, $\rho(v_1, \ldots, v_8) = (4, 2, 1)$ and

$$\begin{aligned} &s_8 = 1, \quad s_1 = s_6 = s_7 = 2, \quad s_2 = s_3 = s_4 = 3, \quad s_5 = 9, \\ &a_8 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 2. \end{aligned}$$

Because $J = \{1, 2, 3, 4, 5\}$ is such that

$$\left\lceil \frac{|J|}{\max\{d_1(v_j : j \in J), 1\}} \right\rceil = 3$$

and there is not an $I \subseteq \{1, \ldots, 8\}$ such that $1 \in I$, $(\langle (v_j)_{j\in I} \rangle \cap (v_1, \ldots, v_8)) = (v_j)_{j\in I}$ and

$$\left\lceil \frac{|I|}{\max\{d_1(v_j : j \in I), 1\}} \right\rceil > 3,$$

then $b_1 = 3$.

In the same way we can conclude that, $b_2 = b_3 = b_4 = b_5 = 4,\ b_6 = b_7 = b_8 = 3$.

Remark. For each $i \in \{1, \ldots, m\}$, it is easy to prove that $1 \leqslant a_i \leqslant m$ and $1 \leqslant b_i \leqslant m$.

Theorem 3.5. *Let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$ and let v_i be an element of $v_1, \ldots, v_m$ such that $s_i = s_{v_i}(v_1, \ldots, v_m)$. Then*

$$\mathscr{S}_i(v_1, \ldots, v_m) = \{r \in \mathbb{N} :\ a_i \leqslant r \leqslant b_i\} \cup \{s_i\}.$$

Theorem 3.6. *Let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$ and let v_i be a nonzero vector of $v_1, \ldots, v_m$. Then*

$$\mathscr{S}_i(v_1, \ldots, v_m) = \{r \in \mathbb{N} :\ a_i \leqslant r \leqslant b_i\}.$$

Now we are going to see consequences of these two theorems.

Proposition 3.7. *Let $v_1, \ldots, v_m$ be a family of vectors of $\mathbb{F}^n$. Then*

$$\bigcap_{i=1}^{m} \mathscr{S}_i(v_1, \ldots, v_m) \neq \emptyset.$$

Proposition 3.8. *Let $v_1, \dots, v_m$ be a family of nonzero vectors of $\mathbb{F}^n$ and let $(\rho_1, \dots, \rho_m)$ be the rank partition of $v_1, \dots, v_m$. Then*

$$\bigcup_{i=1}^{m} \mathscr{S}_i(v_1, \dots, v_m) = \{r \in \mathbb{N} : c \leqslant r \leqslant d\},$$

where

$$c = \begin{cases} 1 & \text{if } \rho_1 < n, \\ \min\{s_i : 1 \leqslant i \leqslant m\} & \text{if } \rho_1 = n \end{cases}$$

and

$$d = \text{length of } \rho(v_1, \dots, v_m).$$

Proposition 3.9. *Let $v_1, \dots, v_m$ be a family of vectors of $\mathbb{F}^n$ with $n > 1$. Let v_i be an element of $v_1, \dots, v_m$:*

$$\mathscr{S}_i(v_1, \dots, v_m) = \{1, \dots, m+1\}$$

if and only if there exists $a \in \{0, 1, \dots, m-1\}$ such that $\rho(v_1, \dots, v_m) = (1^a)$ and $s_i = m + 1$.

4. Background on matroid theory

Let S be a nonempty finite set with cardinality m. Let $M = (S, \mathscr{I}(M))$ be a matroid on S [9,10]. The set of parts of S

$$\left\{I_1 \cup \ldots \cup I_k : I_i \in \mathscr{I}(M),\ i = 1, \dots, k\right\}$$

is the set of independents of a matroid on S, called the kth power of M. We denote this matroid by $M^{(k)}$ [10].

A k-basis of M is a basis of $M^{(k)}$ and a subset of S is k-independent in M if it is independent in $M^{(k)}$.

An element $x \in S$ is a loop of M if x does not belong to any basis of M and x is a coloop of M if x belongs to every basis of M. The set of loops of M will be denoted by $\mathscr{L}(M)$ and the set of coloops of M is denoted by $\mathscr{C}\mathscr{L}(M)$.

Let $A \subseteq S$. The restriction of M to A is denoted by $M|A$ or by $M\backslash(S\backslash A)$. The closure of A is denoted by $\mathrm{cl}_M(A)$ or simply by $\mathrm{cl}(A)$, if there is no ambiguity to avoid. The rank of A is denoted by $r_M(A)$ or simply by $r(A)$. It is known [2] that the sequence

$$\rho(M) = (r_{M^{(1)}}(S) - r_{M^{(0)}}(S), \dots, r_{M^{(m)}}(S) - r_{M^{(m-1)}}(S)),$$

where $r_{M^{(0)}}(S) = 0$, is a partition of $|S\backslash\mathscr{L}(M)|$ called the rank partition of M. The least nonnegative integer c such that $r_{M^{(c)}}(S) = |S\backslash\mathscr{L}(M)|$, is called the covering number of M. If c is the covering number of M and $\rho(M) = (\rho_1, \dots, \rho_m)$, then

$\rho_{c+1} = \cdots = \rho_m = 0$ and we can write $\rho(M) = (\rho_1, \ldots, \rho_c)$. Remark that the covering number of M is the length of $\rho(M)$.

The following theorem is contained in [3].

Theorem 4.1. *If B is a k-basis of M, there exist k pairwise disjoint independent sets of M, $B_1, \ldots, B_k$ such that*
(i) *$B_1 \cup \cdots \cup B_t$ is a t-basis of M, $t = 1, \ldots, k$;*
(ii) *$B_1 \cup \cdots \cup B_k = B$.*
Moreover, $\mathrm{cl}_M(B_1) \supseteq \cdots \supseteq \mathrm{cl}_M(B_k) \supseteq (S \backslash B)$.

Let B be a k-basis of M. A union $B_1 \cup \cdots \cup B_k$ where $B_1, \ldots, B_k$ satisfy the conditions of Theorem 4.1, is called a k-factorization of B.

If c is the covering number of M then $S \backslash \mathscr{L}(M)$ is a c-basis of M. Remark that if $\rho(M) = (\rho_1, \ldots, \rho_c)$ and $B_1 \cup \cdots \cup B_c$ is a c-factorization of $S \backslash \mathscr{L}(M)$, then

$$|B_j| = \rho_j \quad \text{and} \quad r_{M^{(j)}}(S) = \sum_{i=1}^{j} \rho_i, \quad j = 1, \ldots, c.$$

A subset T of S is an r-transversal of M if there exists a family of r-pairwise disjoint subsets of T, $I_1, \ldots, I_r$ satisfying:
(i) $T = I_1 \cup \cdots \cup I_r$;
(ii) I_i is a basis of $M|T$, $i = 1, \ldots, r$.

It is easy to conclude that T is an r-transversal if and only if T is r-independent and $|T| = r \cdot r_M(T)$.

Remark that if $r > 1$, T is an r-transversal and $I_1 \cup \ldots \cup I_r$ is an r-factorization of T in $M|T$, then $I_1 \cup \cdots \cup I_k$, where $1 \leqslant k < r$, is a k-transversal of M.

Lemma 4.2 [1].
(1) *If T is an r-transversal of M, then*

$$\mathrm{cl}_M(T) = \mathrm{cl}_{M^{(r)}}(T).$$

(2) *If T_1 and T_2 are maximal r-transversals of M (for the inclusion order), then*

$$\mathrm{cl}(T_1) = \mathrm{cl}(T_2).$$

(3) *If C is a circuit of $M^{(r)}$ and $y \in C$, then $C \backslash y$ is an r-transversal of M.*

Theorem 4.3 [1]. *Let M be a matroid on S and let B be an r-basis of M. The following hold*:
(1) *If T_1 and T_2 are r-transversals of M contained in B, then $T_1 \cup T_2$ is an r-transversal. The set of r-transversals contained in B has a maximum for the inclusion order.*
(2) *If T is the maximum r-transversal contained in B, then T is a maximal r-transversal of M and $S \backslash B = \mathrm{cl}(T) \backslash T$.*

Theorem 4.4 [4]. *Let T be an r-transversal contained in an r-basis B of M. Assume that $B_1, \ldots, B_r$ are independent pairwise disjoint subsets of B whose union is B. Then*

$$\mathrm{cl}_M(T \cap B_1) = \cdots = \mathrm{cl}_M(T \cap B_r) = \mathrm{cl}_M(T).$$

Let $x \in (S\backslash \mathscr{L}(M))$. The covering number of x in M is the smallest positive integer s such that $x \in \mathscr{C}\mathscr{L}(M^{(s)})$ [3]. We denote this integer by $s_x(M)$. If $x \in \mathscr{L}(M)$, we define $s_x(M) = |S| + 1$.

It is easy to conclude that s is the covering number of $x \in (S\backslash \mathscr{L}(M))$ if and only if s is the least integer such that $r_{M^{(s)}}(S) > r_{(M\backslash x)^{(s)}}(S\backslash x)$.

Remark that if c is the covering number of M and $B_1 \cup \cdots \cup B_c$ is a c-factorization of a c-basis of M, then $s_x(M) = c$, $\forall x \in B_c$.

The following theorem was proved in [3]:

Theorem 4.5. *For $x \in (S\backslash \mathscr{L}(M))$, $s_x(M) > k$ if and only if there exists a k-transversal, T, of M such that $x \in \mathrm{cl}_M(T)\backslash T$.*

Theorem 4.6 [7]. *Let B be a k-basis of M and let x be an element of S such that $s_x(M) = s \leqslant k$. Then there exists a k-factorization of B, $B_1 \cup \cdots \cup B_k$ such that $x \in B_s$.*

Corollary 4.7 [7]. *Let x, y be elements of S such that $s_x(M) = r \neq s = s_y(M)$. Let B be a k-basis of M, $k \geqslant \max\{r, s\}$. Then there exists a k-factorization of B, $B_1 \cup \cdots \cup B_k$ such that $x \in B_r$ and $y \in B_s$.*

Proposition 4.8. *Let M be a matroid on S with covering number c. Let $\rho(M) = (\rho_1, \ldots, \rho_c)$. If $\rho_i < \rho_{i-1}$, then there exists $x \in S$ such that $i - 1 \leqslant s_x(M) \leqslant i$.*

Proof. Let $i \in \{2, \ldots, c+1\}$ such that $\rho_i < \rho_{i-1}$. First we are going to prove that there exists $x \in S$ such that $s_x(M) \geqslant i - 1$.

If $i - 1 = 1$, by definition $s_x(M) \geqslant 1 = i - 1$, $\forall x \in S$. Suppose that $i - 1 > 1$. Let B be an i-basis of M and let $B_1 \cup \cdots \cup B_{i-2} \cup B_{i-1} \cup B_i$ be an i-factorization of B. Then, $|B_{i-1}| = \rho_{i-1} > 0$. Let T be the maximum $(i-2)$-transversal contained in $B_1 \cup \cdots \cup B_{i-2}$. By Theorem 4.3,

$$(B_{i-1} \cup B_i) \subseteq \mathrm{cl}_M(T)\backslash T.$$

By Theorem 4.5,

$$s_x(M) \geqslant i - 1 \quad \forall x \in (B_{i-1} \cup B_i).$$

Now we are going to see that there exists $x \in B_{i-1}$ such that $s_x(M) \leqslant i$.

Let $B_{i-1} = \{x_{i_1}, \ldots, x_{i_p}\}$ and suppose that $s_x(M) > i$ $\forall x \in B_{i-1}$. For each $j \in \{1, \ldots, p\}$, let $y_{i_j} \in S\backslash B$ such that $(B\backslash x_{i_j}) \cup y_{i_j}$ is an i-basis of $M\backslash x_{i_j}$. For each $j \in \{1, \ldots, p\}$, let $C_j = C(B, y_{i_j})$ be the fundamental circuit of y_{i_j} in B (circuit of $M^{(i)}$). Then, for each $j \in \{1, \dots, p\}$, $\{x_{i_j}, y_{i_j}\} \subseteq C_j$. By Lemma 4.2, for each $j \in \{1, \dots, p\}$, $C_j \backslash y_{i_j}$ is an *i*-transversal of *M*, contained in *B*. Using Theorem 4.3,

$$P = \bigcup_{j=1}^{p} (C_j \backslash y_{i_j})$$

is an *i*-transversal of *M*, contained in *B*. Since $B_{i-1} \subseteq P$, using Theorem 4.4, we have $\mathrm{cl}_M(B_i \cap P) = \mathrm{cl}_M(P) \supseteq B_{i-1}$. Consequently, $B_{i-1} \subseteq \mathrm{cl}_M(B_i \cap P) \subseteq \mathrm{cl}_M(B_i)$ and $\rho_{i-1} = |B_{i-1}| \leqslant |B_i| = \rho_i$. But this contradicts the hypothesis, $\rho_i < \rho_{i-1}$. Therefore, there exists $x \in B_{i-1}$ such that $s_x(M) \leqslant i$ and we get the result. □

Let *I* be a nonempty finite set and let $(v_i)_{i\in I}$ be a family of vectors of $\mathbb{F}^n$. We say that the matroid *M* on *I* is the vectorial matroid of $(v_i)_{i\in I}$ if $A \subseteq I$ is independent in *M* if and only if $(v_i)_{i\in A}$ is linearly independent in $\mathbb{F}^n$. We denote this matroid by $M[I]$.

Let $(v_i)_{i\in I}$ be a family of vectors of $\mathbb{F}^n$ and let $(v_i)_{i\in J}$ be a subfamily of $(v_i)_{i\in I}$. Let $M[I]$ be the vectorial matroid of $(v_i)_{i\in I}$. It follows that the matroid of $(v_i)_{i\in J}$, $M[J]$, is the restriction of $M[I]$ to *J*.

Remark. Let $v_1, \dots, v_m$ be a family of vectors of $\mathbb{F}^n$ and let $M = M[\{1, \dots, m\}]$ be the vectorial matroid of $v_1, \dots, v_m$. Clearly:

- A subfamily $(v_i)_{i\in J}$ is *k*-independent if and only if *J* is *k*-independent in *M*.
- A subfamily $(v_i)_{i\in J}$ is a *k*-basis if and only if *J* is a *k*-basis of *M*.
- The rank partition of $v_1, \dots, v_m$ ($\rho(v_1, \dots, v_m)$) coincides with the rank partition of *M*.
- A subfamily $(v_i)_{i\in T}$ is an *r*-transversal (a maximal *r*-transversal) of $v_1, \dots, v_m$ if and only if *T* is an *r*-transversal (respectively, a maximal *r*-transversal) of *M*.
- The covering number of $v_j \in (v_i)_{i\in I}$ coincides with the covering number of *j* in *M*.

5. Proofs

First we give some results which are needed for the main results.

Proposition 5.1. *For each* $i \in \{1, \dots, m\}$, $a_i \leqslant b_i$.

Proof. Suppose that there exists $i \in \{1, \dots, m\}$ such that $1 \leqslant b_i < a_i$. By definition of a_i, $\rho_1 = \cdots = \rho_{b_i} = n$ and $b_i < a_i \leqslant s_i = s_{v_i}(v_1, \dots, v_m)$.

Let $(v_j)_{j\in B}$ be a b_i-basis of $v_1, \dots, v_m$ such that $v_i \notin (v_j)_{j\in B}$. Consequently, $m > |B| = b_i n$ and $d_1(v_j : j \in B) = n = d_1(v_1, \dots, v_m)$. Since $v_i \in (\langle v_1, \dots, v_m\rangle \cap (v_1, \dots, v_m)) = (v_1, \dots, v_m)$ then

$$\{1, \dots, m\} \in \left\{J : i \in J \subseteq \{1, \dots, m\},\ (\langle (v_j)_{j\in J}\rangle \cap (v_j)_{j\in\{1,\dots,m\}}) = (v_j)_{j\in J}\right\}.$$

Thus we have a contradiction,

$$b_i = \left\lceil \frac{|B|}{d_1(v_j : j \in B)} \right\rceil < \left\lceil \frac{|\{1, \ldots, m\}|}{d_1(v_1, \ldots, v_m)} \right\rceil = \left\lceil \frac{m}{n} \right\rceil \leqslant b_i.$$

Therefore, $a_i \leqslant b_i$. □

Proposition 5.2. *For each $i \in \{1, \ldots, m\}$, if $s_i \leqslant m$ then $a_i \leqslant s_i \leqslant b_i$.*

Proof. By definition of a_i, we have $a_i = \min(\min\{j : \rho_j < n\}, s_i) \leqslant s_i$. Consequently we only have to prove that $s_i \leqslant b_i$.

If $s_i = 1$, then $s_i = 1 \leqslant b_i$. Assume that $s_i > 1$. Since $s_i \leqslant m$, by Theorem 4.5, there exists an $(s_i - 1)$-transversal, $(v_j)_{j \in T}$, of $(v_1, \ldots, v_m)$, such that $v_i \in \langle v_j : j \in T\rangle \backslash (v_j : j \in T)$. Let R be the subset of $\{1, \ldots, m\}$ satisfying $(\langle v_j : j \in T\rangle \cap (v_1, \ldots, v_m)) = (v_j : j \in R)$. Then $i \in R \backslash T$ and $\langle v_j : j \in T\rangle = \langle v_j : j \in R\rangle$. But $d_1(v_j : j \in R) = d_1(v_j : j \in T)$. Consequently,

$$s_i - 1 = \left\lceil \frac{|T|}{d_1((v_j)_{j \in T})} \right\rceil < s_i \leqslant \left\lceil \frac{|R|}{d_1((v_j)_{j \in R})} \right\rceil \leqslant b_i.$$

So, $a_i \leqslant s_i \leqslant b_i$. □

Proposition 5.3. $\min\{a_i : 1 \leqslant i \leqslant m\} = \begin{cases} 1 & \text{if } \rho_1 < n, \\ \min\{s_i : 1 \leqslant i \leqslant m\} & \text{if } \rho_1 = n. \end{cases}$

Proof. If $\rho_1 < n$, then by definition of a_i, $a_i = 1$ for every $i \in \{1, \ldots, m\}$. So

$$\min\{a_i : 1 \leqslant i \leqslant m\} = 1.$$

Assume that $\rho_1 = n$. We are going to prove that

$$\min\{a_i : 1 \leqslant i \leqslant m\} = \min\{s_i : 1 \leqslant i \leqslant m\}.$$

Let $1 < j \leqslant m + 1$ be the least integer such that $\rho_j < n$. So $\rho_1 = \cdots = \rho_{j-1} = n$. Since $\rho_j < \rho_{j-1}$, by Proposition 4.8 there exists v_i in $v_1, \ldots, v_m$ such that $s_i \leqslant j$. Then, we can conclude

$$\begin{aligned} \min\{a_i : 1 \leqslant i \leqslant m\} &= \min(\min\{s_i : 1 \leqslant i \leqslant m\}, \min\{k : \rho_k < n\}) \\ &= \min\{s_i : 1 \leqslant i \leqslant m\}. \end{aligned}$$

So we get the result. □

Proposition 5.4.

$$\max\{a_i : 1 \leqslant i \leqslant m\} = \begin{cases} a & \text{if } \rho(v_1, \ldots, v_m) = (n^a) \text{ and } an = m, \\ \min\{j : \rho_j < n\} & \text{otherwise.} \end{cases}$$

Proof. If $\rho(v_1, \ldots, v_m) = (n^a)$ and $an = m$, then $\rho_{a+1} = 0$ and $v_1, \ldots, v_m$ is an a-tranversal of $v_1, \ldots, v_m$. It is easy to calculate s_i and deduce that $s_i = a$ for every $i \in \{1, \ldots, m\}$. Therefore, $a_i = a$, for every $i \in \{1, \ldots, m\}$ and consequently, $\max\{a_i : 1 \leqslant i \leqslant m\} = a$. Suppose there is not $a \in \mathbb{N}$ such that $\rho(v_1, \ldots, v_m) = (n^a)$ and $an = m$. Since $a_i \leqslant \min\{j : \rho_j < n\}$, for every $i \in \{1, \ldots, m\}$, then $\max\{a_i : 1 \leqslant i \leqslant m\} \leqslant \min\{j : \rho_j < n\}$. Let $k = \min\{j : \rho_j < n\}$. Then $\rho_1 = \cdots = \rho_{k-1} = n$. Suppose that $a_i < k$ for every $i \in \{1, \ldots, m\}$. Then $s_i < k \leqslant m$, for every $i \in \{1, \ldots, m\}$. Consequently, $v_1, \ldots, v_m$ is a $(k-1)$-basis of $v_1, \ldots, v_m$. Hence $\rho(v_1, \ldots, v_m) = (n^a)$ with $a = k - 1$. But this contradicts the hypothesis. So there exists $1 \leqslant i \leqslant m$ such that $a_i = k$. Therefore, $\max\{a_i : 1 \leqslant i \leqslant m\} \geqslant k = \min\{j : \rho_j < n\}$ and we get the result. □

Proposition 5.5. $\max\{b_i :\ 1 \leqslant i \leqslant m\} \geqslant$ *length of* $\rho(v_1, \ldots, v_m)$.

Proof. Let c be the length of $\rho(v_1, \ldots, v_m)$. Suppose $c \leqslant 1$. Since, $b_i \geqslant 1$, for every $i \in \{1, \ldots, m\}$, then $\max\{b_i : 1 \leqslant i \leqslant m\} \geqslant c$.

Suppose that $c > 1$. Let v_j be an element of $v_1, \ldots, v_m$ such that $s_j = c$. By Theorem 4.5, there exists a $(c-1)$-transversal, $(v_k)_{k\in T}$, such that $v_j \in \langle (v_k)_{k\in T}\rangle \backslash (v_k)_{k\in T}$. Let $R \subseteq \{1, \ldots, m\}$ satisfying $(\langle (v_k)_{k\in T}\rangle \cap (v_k)_{k\in\{1,\ldots,m\}}) = (v_k)_{k\in R}$. Then

$$\max\{b_i : i \in \{1, \ldots, m\}\} \geqslant b_j \geqslant \left\lceil \frac{|R|}{d_1((v_k)_{k\in R})} \right\rceil > \left\lceil \frac{|T|}{d_1((v_k)_{k\in T})} \right\rceil = c - 1.$$

So we get the result. □

Proposition 5.6. *If* $v_1, \ldots, v_m$ *are nonzero vectors of* $\mathbb{F}^n$, *then*

$$\max\{b_i :\ 1 \leqslant i \leqslant m\} = \text{length of } \rho(v_1, \ldots, v_m).$$

Proof. Let c be the length of $\rho(v_1, \ldots, v_m)$. By Proposition 5.5 we have that $\max\{b_i : 1 \leqslant i \leqslant m\} \geqslant c$. Now we are going to prove that

$$\max\{b_i : 1 \leqslant i \leqslant m\} \leqslant c.$$

If $c = 1$, then $v_1, \ldots, v_m$ is a basis of $v_1, \ldots, v_m$. So, if $J \subseteq \{1, \ldots, m\}$, then $d_1(v_k : k \in J) = |J|$. Consequently, $b_i = 1$, for every $i \in \{1, \ldots, m\}$. Therefore, $\max\{b_i : 1 \leqslant i \leqslant m\} = c$.

Suppose that $c > 1$. Let $f = \max\{b_i : 1 \leqslant i \leqslant m\}$ and consider v_j and $(v_k)_{k\in I}$ such that $j \in I \subseteq \{1, \ldots, m\}$, $(\langle v_k : k \in I\rangle \cap (v_1, \ldots, v_m)) = (v_k : k \in I)$ and $\lceil |I|/d_1((v_k)_{k\in I})\rceil = f$. Let $J \subseteq I$ such that $(v_k)_{k\in J}$ is a maximal $(f-1)$-transversal contained in $(v_k)_{k\in I}$ (remark that $f = \max\{b_i : 1 \leqslant i \leqslant m\} \geqslant c > 1$). By Theorem 4.3, $(v_k)_{k\in I\backslash J} \subseteq \langle (v_k)_{k\in J}\rangle$. Let $l \in I\backslash J$. By Theorem 4.5, $f \leqslant s_l$. Since $s_l \leqslant c$, then $\max\{b_i : 1 \leqslant i \leqslant m\} = f \leqslant c$. Consequently we get the result. □

Proposition 5.7.

$$\min\{b_i : 1 \leqslant i \leqslant m\} \geqslant \left\lceil \frac{m}{\max\{d_1(v_1, \ldots, v_m), 1\}} \right\rceil.$$

Proof. Since for each $i \in \{1, \ldots, m\}$, $(\langle v_1, \ldots, v_m\rangle \cap (v_1, \ldots, v_m)) = (v_1, \ldots, v_m)$ then

$$b_i \geqslant \left\lceil \frac{|\{1, \ldots, m\}|}{\max\{d_1(v_1, \ldots, v_m), 1\}} \right\rceil = \left\lceil \frac{m}{\max\{d_1(v_1, \ldots, v_m), 1\}} \right\rceil. \qquad \square$$

Proposition 5.8. *Let v_i be a nonzero vector of $v_1, \ldots, v_m$ such that $a_i = b_i = s_i$. If there exists an s_i-transversal, $(v_j)_{j \in J}$, such that $i \in J \subseteq \{1, \ldots, m\}$ then*

$$s_i - 1 \leqslant s_j \leqslant s_i \quad \forall j \in \{1, \ldots, m\}.$$

Proof. First we are going to see that $s_j \geqslant s_i - 1$, for every $j \in \{1, \ldots, m\}$. If $s_i \leqslant 2$, we have $s_j \geqslant 1 \geqslant s_i - 1$, for every $j \in \{1, \ldots, m\}$.

Suppose that $s_i > 2$. By definition of a_i, $\rho_1 = \ldots = \rho_{s_i - 1} = n$. Consequently, an $(s_i - 1)$-basis of $v_1, \ldots, v_m$ is a maximal $(s_i - 1)$-transversal of $v_1, \ldots, v_m$. So by Theorem 4.5, $s_j \geqslant s_i - 1$, for every $j \in \{1, \ldots, m\}$.

Now we are going to prove that $s_j \leqslant s_i$, for every $j \in \{1, \ldots, m\}$. Suppose there exists v_j in $v_1, \ldots, v_m$ such that $s_j > s_i$. Let $(v_k)_{k \in B}$ be an (s_j)-basis of $v_1, \ldots, v_m$. By Corollary 4.7, let

$$(v_k)_{k \in B_1} \cup \cdots \cup (v_k)_{k \in B_{s_i}} \cup \cdots \cup (v_k)_{k \in B_{s_j}}$$

be an (s_j)-factorization of $(v_k)_{k \in B}$ such that $v_i \in B_{s_i}$ and $v_j \in B_{s_j}$. Let $(v_k)_{k \in R}$ be the maximum s_i-transversal contained in $(v_k)_{k \in B_1} \cup \cdots \cup (v_k)_{k \in B_{s_i}}$. By hypothesis and by Lemma 4.2, we can say that $v_i \in (v_k)_{k \in J} \subseteq \langle v_k : k \in R\rangle$. Since $s_i(v_1, \ldots, v_m) = s_i$, by Theorem 4.3 we can conclude that $v_i \in (v_k)_{k \in R}$. On the other hand, since $v_j \in B_{s_j}$, using Theorem 4.3, $v_j \in \langle v_k : k \in R\rangle \backslash (v_k : k \in R)$. Let $(v_k)_{k \in P} = (\langle v_k : k \in R\rangle \cap (v_1, \ldots, v_m))$. So we have $v_i, v_j \in (v_k)_{k \in P}$, $d_1(v_k : k \in P) = d_1(v_k : k \in R)$ and

$$\left\lceil \frac{|P|}{\max\{d_1(v_k : k \in P), 1\}} \right\rceil \geqslant \left\lceil \frac{|R| + 1}{\max\{d_1(v_k : k \in R), 1\}} \right\rceil$$
$$\geqslant \left\lceil s_i + \frac{1}{\max\{d_1(v_k : k \in R), 1\}} \right\rceil = s_i + 1.$$

But this contradicts the fact that

$$\left\lceil \frac{|P|}{\max\{d_1(v_k : k \in P), 1\}} \right\rceil \leqslant b_i = s_i.$$

So $s_j \leqslant s_i$, for every $j \in \{1, \ldots, m\}$ and we get the result. $\square$

Example 5.9. The converse of Proposition 5.8 does not hold, as the following example shows. Let e_1, e_2, e_3 be a basis of $\mathbb{F}^3$. Let $m = 5$ and define

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_1 + e_2, \quad v_4 = e_3, \quad v_5 = e_1 + e_3.$$

Then, $s_1 = \cdots = s_5 = 2$, $a_1 = \cdots = a_5 = 2$ and $b_1 = \cdots = b_5 = 2$ but there is not a 2-transversal of $v_1, \ldots, v_5$, different from the empty one. □

Proposition 5.10. *There exists an* $1 \leqslant i \leqslant m$ *such that* $a_i = b_i = s_i = 1$ *if and only if* $s_j = 1\ \forall j \in \{1, \ldots, m\}$.

Proof. If there exists $1 \leqslant i \leqslant m$ such that $a_i = b_i = s_i = 1$, by Proposition 5.8, $s_j = 1\ \forall j \in \{1, \ldots, m\}$.

If $s_j = 1\ \forall j \in \{1, \ldots, m\}$, then $a_j = 1$ and $v_1, \ldots, v_m$ is a basis of $v_1, \ldots, v_m$. Consequently, if $J \subseteq \{1, \ldots, m\}$, $d_1((v_k)_{k\in J}) = |J|$. Then $\left\lceil \frac{|J|}{d_1((v_j)_{j\in J})} \right\rceil = 1$. So, $b_j = 1$ and there exists $1 \leqslant i \leqslant m$ such that $s_i = a_i = b_i = 1$. □

The following lemma was proved in [8].

Lemma 5.11. *Let* S *be a subspace of* $\mathbb{F}^n$. *Let* $u_1, \ldots, u_p, u_{p+1}, \ldots, u_q \in S$. *If the family* $u_1, \ldots, u_p$ *is linearly independent and* $q \leqslant \dim S$, *then, for every* $\delta > 0$, *there exist* $u'_{p+1}, \ldots, u'_q \in S$ *such that*

$$\|u'_j - u_j\| \leqslant \delta, \quad j \in \{p+1, \ldots, q\}$$

and

$$(u_1, \ldots, u_p, u'_{p+1}, \ldots, u'_q) \text{ is linearly independent.}$$

Lemma 5.12. *If* $a_i \leqslant r \leqslant b_i$ *and* $r < s_i$, *then* $r \in \mathscr{S}_i(v_1, \ldots, v_m)$.

Proof. Let $(\rho_1, \ldots, \rho_m)$ be the rank partition of $v_1, \ldots, v_m$. By hypothesis, $s_i > r \geqslant a_i = \min(\min\{j : \rho_j < n\}, s_i)$, then $\min\{j : \rho_j < n\} < s_i$. Consequently, $s_i > 1$ and $\rho_r < n$.

Let $(v_j)_{j\in B}$ be an r-basis of $v_1, \ldots, v_m$ such that v_i does not belong and let

$$(v_j)_{j\in B_1} \cup \cdots \cup (v_j)_{j\in B_r}$$

be an r-factorization of $(v_j)_{j\in B}$.

Let $\delta > 0$. We are going to prove that there exist $v'_1, \ldots, v'_m \in \mathbb{F}^n$ such that

$$\|v'_j - v_j\| \leqslant \delta, \quad j \in \{1, \ldots, m\} \text{ and } s_{v'_i}(v'_1, \ldots, v'_m) = r.$$

1. Suppose that $|\{1, \ldots, m\} \backslash (B \cup \{i\})| \geqslant (r-1)|B_1| - |B_1 \cup \cdots \cup B_{r-1}| = s$. Since, for every $u \in \{1, \ldots, r-1\}$, $\langle v_k : k \in B_u\rangle$ is a subspace of $\langle v_k : k \in B_1\rangle$, using Lemma 5.11, there exist $(v'_j)_{j\in T}$, with $T \subseteq \{1, \ldots, m\} \backslash (B \cup \{i\})$ such that
   1. $|T| = s$,
   2. $v'_j \in \langle v_k : k \in B_1\rangle\ \forall j \in T$,
   3. $||v'_j - v_j|| \leqslant \delta\ \forall j \in T$,

4. $((v_j)_{j\in B_1\cup\cdots\cup B_{r-1}} \cup (v'_j)_{j\in T})$ is an $(r-1)$-basis of $((v_j)_{j\in\{1,\ldots,m\}\backslash T} \cup (v'_j)_{j\in T})$.

Since $\langle v_k : k \in B_r \cup \{i\}\rangle \subseteq \langle v_k : k \in B_1\rangle$ and $|B_r| + 1 \leqslant n$, again by Lemma 5.11, there exists v'_i such that

1. $v'_i \in \langle v_j : j \in B_1\rangle$,
2. $\|v'_i - v_i\| \leqslant \delta$,
3. $((v_j)_{j\in B_r} \cup (v'_i))$ is linearly independent.

Then,

$$|B_1 \cup \cdots \cup B_{r-1} \cup T| = (r-1)|B_1|.$$

This implies that $((v_j)_{j\in B_1\cup\cdots\cup B_{r-1}} \cup (v'_j)_{j\in T})$ is an $(r-1)$-basis of $((v_j)_{j\in\{1,\ldots,m\}\backslash(T\cup\{i\})} \cup (v'_j)_{j\in T\cup\{i\}})$ where v'_i does not belong. Consequently,

$$s_{v'_i}((v_j)_{j\in\{1,\ldots,m\}\backslash(T\cup\{i\})} \cup (v'_j)_{j\in T\cup\{i\}}) \geqslant r.$$

On the other hand, since $(v_j)_{j\in B_1\cup\cdots\cup B_r}$ is an r-basis of $v_1,\ldots,v_m$ then $((v_j)_{j\in B_1\cup\cdots\cup B_r} \cup (v'_j)_{j\in T\cup\{i\}})$ is an r-basis of $((v_j)_{j\in\{1,\ldots,m\}\backslash(T\cup\{i\})} \cup (v'_j)_{j\in T\cup\{i\}})$ and $((v_j)_{j\in B_1\cup\cdots\cup B_r} \cup (v'_j)_{j\in T})$ is an r-basis of $((v_j)_{j\in\{1,\ldots,m\}\backslash(T\cup\{i\})} \cup (v'_j)_{j\in T})$. Consequently,

$$s_{v'_i}((v_j)_{j\in\{1,\ldots,m\}\backslash(T\cup\{i\})} \cup (v'_j)_{j\in T\cup\{i\}}) = r.$$

2. Now suppose that $|\{1,\ldots,m\}\backslash(B\cup\{i\})| < (r-1)|B_1| - |B_1\cup\cdots\cup B_{r-1}| = s$. If v_i is a nonzero vector of $\mathbb{F}^n$, let $C = C(B_r, i)$ be the fundamental circuit of i in B_r and let l be an element of $C\backslash\{i\}$. If v_i is the zero vector of $\mathbb{F}^n$, let l be an element of B_r. Since for every $u \in \{1,\ldots,r-1\}$, $\langle v_k : k \in B_u\rangle$ is a subspace of $\langle v_k : k \in B_1\rangle$, using Lemma 5.11, there exist $(v'_j)_{j\in\{1,\ldots,m\}\backslash(B\cup\{i\})}$ and v'_l such that
   1. $v'_j \in \langle v_k : k \in B_1\rangle \quad, \forall j \in (\{1,\ldots,m\}\backslash(B\cup\{i\})) \cup \{l\}$
   2. $||v'_j - v_j|| \leqslant \delta \quad, \forall j \in (\{1,\ldots,m\}\backslash(B\cup\{i\})) \cup \{l\}$
   3. $((v_j)_{j\in B_1\cup\ldots\cup B_{r-1}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash(B\cup\{i\}))\cup\{l\}})$ is an $(r-1)$-basis of $((v_j)_{j\in(B\backslash\{l\})\cup\{i\}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash(B\cup\{i\}))\cup\{l\}})$.

   Since $\langle v_k : k \in (B_r\backslash\{l\}) \cup \{i\}\rangle \subseteq \langle v_k : k \in B_r\rangle$, again by Lemma 5.11, there exists v'_i such that
   1. $v'_i \in \langle v_j : j \in B_r\rangle$
   2. $||v'_i - v_i|| \leqslant \delta$
   3. $((v_j)_{j\in B_r\backslash\{l\}} \cup (v'_i))$ is linearly independent.

   Let $(v_j)_{j\in T}$ be a maximum $(r-1)$-transversal of $v_1,\ldots,v_m$ contained in $(v_j)_{j\in B_1\cup\ldots\cup B_{r-1}}$. Since $v'_i \in \langle v_j : j \in B_r\rangle \subseteq \langle v_j : j \in T\rangle$, then $v'_i \in \langle v_j : j \in T\rangle\backslash(v_j : j \in T)$. But $(v_j)_{j\in T}$ is an $(r-1)$-transversal of $((v_j)_{j\in B\backslash\{l\}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash B)\cup\{l\}})$, then

   $$s_{v'_i}((v_j)_{j\in B\backslash\{l\}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash B)\cup\{l\}}) \geqslant r.$$

   Because $((v_j)_{j\in B\backslash\{l\}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash B)\cup\{l\}})$ is an r-basis of $((v_j)_{j\in B\backslash\{l\}} \cup (v'_j)_{j\in(\{1,\ldots,m\}\backslash B)\cup\{l\}})$ then

$$s_{v_i'}((v_j)_{j\in B\backslash\{l\}} \cup (v_j')_{j\in(\{1,\dots,m\}\backslash B)\cup\{l\}}) = r.$$

So, $r \in \mathscr{S}_i(v_1, \dots, v_m)$. $\square$

Lemma 5.13. *If $a_i \leqslant r \leqslant b_i$ and $r > s_i$ then $r \in \mathscr{S}_i(v_1, \dots, v_m)$.*

Proof. By hypothesis, $r > s_i$ and $a_i \leqslant r \leqslant b_i \leqslant m$, so $s_i \neq m+1$ and v_i is a nonzero vector of $\mathbb{F}^n$. Consequently, if $i \in R \subseteq \{1, \dots, m\}$, $\max\{d_1(v_j : j \in R), 1\} = d_1(v_j : j \in R)$.

Let $\delta > 0$. We are going to prove that there exist $v_1', \dots, v_m' \in \mathbb{F}^n$ such that

$$||v_j' - v_j|| \leqslant \delta, \quad j \in \{1, \dots, m\} \text{ and } s_{v_i'}(v_1', \dots, v_m') = r.$$

Let

$$\mathscr{R} = \left\{ R : i \in R \subseteq \{1, \dots, m\},\ (\langle (v_j)_{j\in R}\rangle \cap (v_j)_{j\in\{1,\dots,m\}}) \right.$$
$$\left. = (v_j)_{j\in R},\ \left\lceil \frac{|R|}{d_1((v_j)_{j\in R})} \right\rceil \geqslant r \right\}.$$

Claim 1. $\mathscr{R} \neq \emptyset$.

Proof. Let $A \subseteq \{1, \dots, m\}$ such that $i \in A$, $(\langle (v_j)_{j\in A}\rangle \cap (v_j)_{j\in\{1,\dots,m\}}) = (v_j)_{j\in A}$ and $\left\lceil \frac{|A|}{d_1((v_j)_{j\in A})} \right\rceil = b_i$. Since $b_i \geq r$, then $A \in \mathscr{R}$ and consequently $\mathscr{R} \neq \emptyset$. $\square$

Let P be an element of $\mathscr{R}$ such that

$$((r-1)d_1((v_j)_{j\in P}) - d_{r-1}((v_j)_{j\in P})$$
$$= \min\{((r-1)d_1((v_j)_{j\in R}) - d_{r-1}((v_j)_{j\in R}) : R \in \mathscr{R}\}.$$

Since $P \in \mathscr{R}$, then

$$\left\lceil \frac{|P|}{d_1((v_j)_{j\in P})} \right\rceil \geqslant r.$$

Then we have $|P| > (r-1)d_1(v_j : j \in P)$. Consequently, $|P\backslash\{i\}| \geqslant (r-1)d_1(v_j : j \in P)$. Let $k = s_{v_i}(v_j : j \in P)$.

Claim 2. $k \leqslant s_i < r$.

Proof. If $k = 1$, by definition of covering number of an element, $s_i \geq 1 = k$. Suppose that $k > 1$. Since v_i is a nonzero vector and $k \leqslant m$, using Theorem 4.5, there exists a $(k-1)$-transversal, $(v_j)_{j\in T}$, of $(v_j)_{j\in P}$ such that $v_i \in \langle v_j : j \in T\rangle\backslash(v_j : j \in T)$. Then $(v_j)_{j\in T}$ is a $(k-1)$-transversal of $(v_j)_{j\in\{1,\dots m\}}$ and $v_i \in \langle v_j : j \in T\rangle\backslash(v_j : j \in T)$. This implies, by Theorem 4.5 that $s_i = s_{v_i}(v_1, \dots, v_m) \geq k$. $\square$

By Theorem 4.6, let

$$(v_j)_{j\in P_1} \cup \ldots \cup (v_j)_{j\in P_k} \cup \ldots \cup (v_j)_{j\in P_r} \cup \ldots \cup (v_j)_{j\in P_m}$$

be an m-factorization of $(v_j)_{j\in P}$ such that $v_i \in (v_j)_{j\in P_k}$. Now we are going to prove that there exist $(v'_j)_{j\in P\backslash\{i\}}$ such that

$$||v'_j - v_j|| \leqslant \delta \quad \forall j \in P\backslash\{i\} \text{ and } s_{v_i}((v'_j)_{j\in P\backslash\{i\}} \cup (v_i)) = r.$$

Since $|P_1 \cup \ldots \cup P_{r-1}| + |P_r \cup \ldots \cup P_m| = |P| > (r-1)d_1(v_j : j \in P)$, let $q \geqslant r$ be the integer such that

$$|P_{q+1} \cup \ldots \cup P_m \cup P_{m+1}| < (r-1)d_1(v_j : j \in P) - |P_1 \cup \ldots \cup P_{r-1}| + 1 = f$$

and

$$|P_q \cup \ldots \cup P_m \cup P_{m+1}| \geqslant (r-1)d_1(v_j : j \in P) - |P_1 \cup \ldots \cup P_{r-1}| + 1 = f$$

where $P_{m+1} = \emptyset$.

Let $P' = P_{q+1} \cup \ldots \cup P_m \cup P_{m+1}$ and $J = P' \cup X$ such that $X \subseteq P_q$ and $|J| = f$.

But for every $u \in \{1, \ldots, r-1\}$, $\langle v_j : j \in P_u\rangle$ is a subspace of $\langle v_j : j \in P_1\rangle$, then by Lemma 5.11, there exist $(v'_j)_{j\in J}$ such that

1. $v'_j \in \langle v_j : j \in P_1\rangle \quad \forall j \in J$
2. $||v'_j - v_j|| \leqslant \delta \quad \forall j \in J$
3. $((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J})$ is an $(r-1)$-basis of $((v_j)_{j\in P\backslash J} \cup (v'_j)_{j\in J})$.

Claim 3. $s_{v_i}((v_j)_{j\in P\backslash J} \cup (v'_j)_{j\in J}) \geqslant r$.

Proof. Since

$$\begin{aligned} d_{r-1}((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J}) &= (r-1)d_1(v_j : j \in P) \\ &= (r-1)d_1((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J}), \end{aligned}$$

then $((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J})$ is an $(r-1)$-transversal of $((v_j)_{j\in P\backslash J} \cup (v'_j)_{j\in J})$ and

$$v_i \notin ((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J}).$$

But $v_i \in \langle v_j : j \in P_1\rangle \subseteq \langle((v_j)_{j\in P_1\cup\ldots\cup P_{r-1}\backslash\{i\}} \cup (v'_j)_{j\in J})\rangle$. So, by Theorem 4.5,

$$\text{cf. } s_{v_i}((v_j)_{j\in P\backslash J} \cup (v'_j)_{j\in J}) \geqslant r. \qquad \square$$

By Theorem 4.5, we can conclude that

$$s_{v_i}((v_j)_{j\in\{1,\ldots,m\}\backslash J} \cup (v'_j)_{j\in J}) \geqslant r.$$

Suppose that

$$s_{v_i}((v_j)_{j\in\{1,\ldots,m\}\backslash J} \cup (v'_j)_{j\in J}) > r$$

and let $T' \subseteq \{1, \ldots, m\}\backslash J$ and $J' \subseteq J$ such that $((v_j)_{j\in T'} \cup (v'_j)_{j\in J'})$ is an r-transversal of $((v_j)_{j\in\{1,\ldots,m\}\backslash J} \cup (v'_j)_{j\in J})$ satisfying

$$v_i \in \langle((v_j)_{j\in T'} \cup (v'_j)_{j\in J'})\rangle\backslash((v_j)_{j\in T'} \cup (v'_j)_{j\in J'}).$$

Consider the following sets

$$Z \subseteq \{1, \ldots, m\}\backslash J,\ Z' \subseteq J \text{ and } W \subseteq \{1, \ldots, m\}$$

such that

$$\begin{aligned}&(\langle((v_j)_{j\in T'} \cup (v'_j)_{j\in J'})\rangle \cap ((v_j)_{j\in\{1,\ldots,m\}\backslash J} \cup (v'_j)_{j\in J})\\ &\quad= ((v_j)_{j\in Z} \cup (v'_j)_{j\in Z'})\end{aligned}$$

and

$$(\langle(v_j)_{j\in Z\cup Z'}\rangle \cap (v_j)_{j\in\{1,\ldots,m\}}) = (v_j)_{j\in W}.$$

Remark that $i \in Z$ and $i \in W$.

Claim 4. $W \in \mathscr{R}$.

Proof. Using Lemma 3.1, we have that

$$\begin{aligned}d_1(v_j : j \in W) &= d_1(v_j : j \in Z \cup Z')\\ &\leqslant d_1((v_j)_{j\in Z} \cup (v'_j)_{j\in Z'})\\ &= d_1((v_j)_{j\in T'} \cup (v'_j)_{j\in J'}).\end{aligned}$$

Since $|W| \geq |Z \cup Z'| \geq |T' \cup J'|$, we can conclude that

$$\left\lceil \frac{|W|}{d_1((v_j)_{j\in W})} \right\rceil \geq \left\lceil \frac{|T' \cup J'|}{d_1((v_j)_{j\in T'} \cup (v'_j)_{j\in J'})} \right\rceil = r.$$

Consequently $W \in \mathscr{R}$. $\square$

Claim 5.

$$\begin{aligned}&((r-1)d_1((v_j)_{j\in P})) - d_{r-1}((v_j)_{j\in P})\\ &\quad> ((r-1)d_1((v_j)_{j\in W})) - d_{r-1}((v_j)_{j\in W}).\end{aligned}$$

Proof. Let

$$((v_j)_{j\in T'_1} \cup (v'_j)_{j\in J'_1}) \cup \ldots \cup ((v_j)_{j\in T'_r} \cup (v'_j)_{j\in J'_r})$$

be an r-factorization of $((v_j)_{j\in T'} \cup (v'_j)_{j\in J'})$, with $T'_k \subseteq T'$ and $J'_k \subseteq J'$, $k = 1, \ldots, r$ and such that $J'_r \cap J' \neq \emptyset$. Therefore,

$$((v_j)_{j\in T'_1} \cup (v'_j)_{j\in J'_1}) \cup \ldots \cup ((v_j)_{j\in T'_{r-1}} \cup (v'_j)_{j\in J'_{r-1}})$$

is an $(r-1)$-transversal of $((v_j)_{j\in\{1,\ldots,m\}\backslash J} \cup (v'_j)_{j\in J})$.

If A is the set $J_1' \cup \ldots \cup J_{r-1}'$ and B is the set $T_1' \cup \ldots \cup T_{r-1}'$, then

$$|A| = |J_1' \cup \ldots \cup J_{r-1}'| < |J| = f.$$

Because $s_i < r$, if $(v_j)_{j\in C}$ is an $(r-1)$-basis of $(v_j)_{j\in W}$, then $v_i \in (v_j)_{j\in C}$. But $(v_j)_{j\in B}$ is $(r-1)$ independent in $(v_j)_{j\in W}$, so $(v_j)_{j\in B\cup\{i\}}$ is $(r-1)$ independent in $(v_j)_{j\in W}$. Since

$$(r-1)d_1((v_j)_{j\in B} \cup (v_j')_{j\in A}) - d_{r-1}((v_j)_{j\in B} \cup (v_j')_{j\in A}) = 0$$

then

$$\begin{aligned} d_{r-1}((v_j)_{j\in W}) &\geqslant d_{r-1}((v_j)_{j\in B}) + 1 \\ &= d_{r-1}((v_j)_{j\in B} \cup (v_j')_{j\in A}) - |A| + 1 \\ &= (r-1)d_1((v_j)_{j\in B} \cup (v_j')_{j\in A}) - |A| + 1 \\ &= (r-1)d_1((v_j)_{j\in T'} \cup (v_j')_{j\in J'}) - |A| + 1. \end{aligned}$$

Consequently we have

$$\begin{aligned} &(r-1)d_1((v_j)_{j\in W}) - d_{r-1}((v_j)_{j\in W}) \\ &\quad\leqslant (r-1)d_1((v_j)_{j\in T'} \cup (v_j')_{j\in J'}) - (r-1)d_1((v_j)_{j\in T'} \cup (v_j')_{j\in J'}) \\ &\qquad + |A| - 1 < |J| - 1 = f - 1 = (r-1)d_1((v_j)_{j\in P}) - d_{r-1}((v_j)_{j\in P}). \end{aligned}$$

□

Since P is an element of $\mathscr{R}$ such that $((r-1)d_1((v_j)_{j\in P}) - d_{r-1}((v_j)_{j\in P}) = \min\{((r-1)d_1((v_j)_{j\in R}) - d_{r-1}((v_j)_{j\in R}) : R \in \mathscr{R}\}$, by Claims 4 and 5 we have a contradiction. So, $s_{v_i}((v_j)_{j\in\{1,\ldots,m\}\setminus J} \cup (v_j')_{j\in J}) = r$ and $r \in \mathscr{S}_i(v_1, \ldots, v_m)$. □

Proof of Theorem 3.5. First we are going to prove that

$$\mathscr{S}_i(v_1, \ldots, v_m) \subseteq \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}.$$

Let $k \in \mathscr{S}_i(v_1, \ldots, v_m)$. If $k = m+1$, by Proposition 3.3 we can conclude that v_i is the zero vector of $\mathbb{F}^n$. Consequently, $s_i = m+1$ and $k = m+1 \in \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}$.

Suppose that $k \neq m+1$. By definition of $\mathscr{S}_i(v_1, \ldots, v_m)$, for every $\delta > 0$, there exist $v_1', \ldots, v_m' \in \mathbb{F}^n$ such that

$$||v_j' - v_j|| \leqslant \delta \quad , j \in \{1, \ldots, m\} \text{ and } s_{v_i'}(v_1', \ldots, v_m') = k.$$

Let $\delta > 0$ and $v_1', \ldots, v_m'$ be a family of vectors of $\mathbb{F}^n$ in this conditions.

We are going to see that $k \leqslant b_i$. If $k = 1$ then $1 \leqslant b_i$. Suppose that $k > 1$. Using Theorem 4.5, there exist a $(k-1)$-transversal, $(v_j')_{j\in T}$, of $(v_j')_{j\in\{1,\ldots,m\}}$ such that

$$v_i' \in \langle (v_j')_{j\in T}\rangle \setminus (v_j')_{j\in T}.$$

Let P be a subset of $\{1, \ldots, m\}$ satisfying $(v_j')_{j\in P} = (\langle (v_j')_{j\in T}\rangle \cap (v_j')_{j\in\{1,\ldots,m\}})$ and let I be a subset of $\{1, \ldots, m\}$ satisfying $(v_j)_{j\in I} = (\langle (v_j)_{j\in P}\rangle \cap (v_j)_{j\in\{1,\ldots,m\}})$.

By Proposition 3.1, $d_1((v_j)_{j\in I}) = d_1((v_j)_{j\in P}) \leqslant d_1((v'_j)_{j\in P}) = d_1((v'_j)_{j\in T})$.

On the other hand, since $T \subseteq P$ and $i \in P\backslash T$ then

$$|T| < |P| \leqslant |I|.$$

Consequently,

$$k-1 = \left\lceil \frac{|T|}{d_1((v'_j)_{j\in T})} \right\rceil < k \leqslant \left\lceil \frac{|I|}{d_1((v_j)_{j\in I})} \right\rceil.$$

Since $i \in P$, then $i \in I$. But $\langle (v_j)_{j\in I}\rangle \cap (v_j)_{j\in\{1,\dots,m\}} = (v_j)_{j\in I}$, so

$$k \leqslant \left\lceil \frac{|I|}{d_1((v_j)_{j\in I})} \right\rceil \leqslant b_i.$$

Now, we are going to prove that $k \geqslant a_i$.

Let $(\rho_1,\dots,\rho_m)$ be the rank partition of $v_1,\dots,v_m$. Suppose that $k < \min(\min\{j : n > \rho_j\}, s_i)$. Since $k \geqslant 1$, then $\rho_1 = \ldots = \rho_k = n$ and consequently $d_k(v_1,\dots,v_m) = kn$. Using Theorem 3.2, $kn = d_k(v_1,\dots,v_m) \leqslant d_k(v'_1,\dots,v'_m) \leqslant \min\{m, kn\}$. Then $d_k(v'_1,\dots,v'_m) = kn$.

Since $s_{v'_i}(v'_1,\dots,v'_m) = k$, then $d_k(v'_1,\dots,v'_{i-1},v'_{i+1},\dots,v'_m) = kn-1$. Again by Theorem 3.2, $d_k(v_1,\dots,v_{i-1},v_{i+1},\dots,v_m) \leqslant kn-1 < d_k(v_1,\dots,v_m)$. This implies that $s_i = s_{v_i}(v_1,\dots,v_m) \leqslant k$. Since $k < s_i$, we have a contradition. Consequently, $k \geq a_i$ and $k \in \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}$. Therefore

$$\mathscr{S}_i(v_1,\dots,v_m) \subseteq \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}.$$

Now we are going to prove that

$$\mathscr{S}_i(v_1,\dots,v_m) \supseteq \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}.$$

Let $k \in \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}$.

If $k = s_i$, then $k \in \mathscr{S}_i(v_1,\dots,v_m)$.

If $k < s_i$, since $a_i \leqslant k \leqslant b_i$ by Lemma 5.12, $k \in \mathscr{S}_i(v_1,\dots,v_m)$.

If $k > s_i$, since $a_i \leqslant k \leqslant b_i$ by Lemma 5.13, $k \in \mathscr{S}_i(v_1,\dots,v_m)$.

Therefore, $\mathscr{S}_i(v_1,\dots,v_m) \supseteq \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}$ and we can conclude that $\mathscr{S}_i(v_1,\dots,v_m) = \{r \in \mathbb{N} : a_i \leqslant r \leqslant b_i\} \cup \{s_i\}$. □

Proof of Theorem 3.6. Consequence of Theorem 3.5 and Proposition 5.2. □

Proof of Proposition 3.7. Let k be the integer such that $s_k = min\{s_j : j \in \{1,\dots,m\}\}$. Let $(\rho_1,\dots,\rho_m)$ be the rank partition of $v_1,\dots,v_m$. By definition of a_i, $a_k = s_k$ or $a_k = min\{j : \rho_j < n\}$.

Suppose that $a_k = \min\{j : \rho_j < n\}$. Then $a_i = a_k$, for every $i \in \{1,\dots,m\}$ and $a_k \in \bigcap_{i=1}^{n} \mathscr{S}_i(v_1,\dots,v_m)$.

Suppose that $a_k = s_k$. Then $\rho_1 = \ldots = \rho_{s_k-1} = n$. If $\rho_{s_k} < n$ or $s_i = s_k$, for every $i \in \{1,\dots,m\}$, using Theorem 3.5, $s_k \in \bigcap_{i=1}^{n} \mathscr{S}_i(v_1,\dots,v_m)$.

If $\rho_{s_k} = n$ and there exists v_j such that $s_j > s_k$, then $1 \leqslant \rho_{s_k+1} < n$ and $a_j = s_k + 1$. Consequently, $a_i \leqslant s_k + 1$, for every $i \in \{1,\dots,m\}$. On the other hand

$$b_i \geq \left\lceil \frac{|\{1,\ldots,m\}|}{d_1((v_j)_{j\in\{1,\ldots,m\}})} \right\rceil = \left\lceil \frac{m}{n} \right\rceil \geq \left\lceil \frac{ns_k+1}{n} \right\rceil = s_k+1 \quad \forall i \in \{1,\ldots,m\}.$$

Then, by Theorem 3.5, $s_k+1 \in \bigcap_{i=1}^{n} \mathscr{S}_i(v_1,\ldots,v_m)$ and $\bigcap_{i=1}^{n} \mathscr{S}_i(v_1,\ldots,v_m) \neq \emptyset$. □

Proof of Proposition 3.8. The proof of this proposition is immediate from Theorem 3.6, Proposition 3.7, Proposition 5.6 and Proposition 5.3. □

Proof of Proposition 3.9. Let $(\rho_1,\ldots,\rho_m)$ be the rank partition of $v_1,\ldots,v_m$. Suppose that $\mathscr{S}_i(v_1,\ldots,v_m) = \{1,\ldots,m+1\}$. Since $m+1 \in \mathscr{S}_i(v_1,\ldots,v_m)$ by Proposition 3.3, $s_i = m+1$. By Theorem 3.5, $a_i = 1$ and $b_i = m$. So, $\rho_1 < n$. If $I \subseteq \{1,\ldots,m\}$, then

$$\left\lceil \frac{|I|}{\max\{d_1((v_j)_{j\in I}),1\}} \right\rceil \leqslant |I|.$$

But $b_i = m$, then $d_1((v_j)_{j\in I}) \leqslant 1$. Consequently, there exists $a \in \{0,\ldots,m\}$ such that $\rho(v_1,\ldots,v_m) = (1^a)$ where a is the number of nonzero vectors in $v_1,\ldots,v_m$.

Now suppose that there exists $a \in \{0,\ldots,m\}$ such that $\rho(v_1,\ldots,v_m) = (1^a)$ and $s_i = m+1$. Since $n > 1$, by definition of a_i, $a_i = 1$. On the other hand, if $I \subseteq \{1,\ldots,m\}$, $\max\{d_1((v_j)_{j\in I}),1\} = 1$. So, $b_i = m$. Using Theorem 3.5, $\mathscr{S}_i(v_1,\ldots,v_m) = \{1,\ldots,m+1\}$. □

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